Thus, our investigation of the nonstationary perturbation spectrum in a vertical channel with permeable boundaries leads to the conclusion that an oscillatory convective instability can exist. For small values of the Péclet number ($a < 0.8$, cf. Fig. 3), as the Rayleigh number increases, transverse motion becomes unstable with respect to monotonic perturbations, i. e. at the critical value of Rayleigh number (curve A) stationary convection begins to take place. For $a > 0.8$ the instability expresses itself in oscillatory perturbations; after crossing the neutral line C (as R increases) an oscillatory convection occurs.

Let us note in conclusion that closure of stationary levels has been detected earlier [3] in a study of convective motion stability in an inclined layer. In the present problem, the closure is accompanied not by stabilization as in [3], but by change in the mode of instability, namely by transition to oscillatory convection.

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APPLICATION OF THE STATIONARY PHASE METHOD IN SOME PROBLEMS OF THE THEORY OF WAVES ON THE SURFACE OF A VISCOUS LIQUID

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The purpose of this paper is to develop the asymptotic representation of certain integrals encountered in the analysis of the problem of wave motion in an unbounded viscous liquid. Attention is also drawn to incorrect application of the stationary phase method widely used in a number of recent publications [2–21] dealing with the Cauchy-Poisson problem of waves on the surface of half-space or layer.

1. Sretenskii [1] published in 1941 a fundamental work on the subject considered in the present paper. The second Chapter of [1] deals with the two-dimensional Cauchy-Poisson problem of waves on the surface of a viscous liquid of infinite depth. By successive integral transformations of Fourier and Laplace he obtained for the first time an exact integral representation for the free surface shape. For the asymptotic calculations of the integrals obtained, the method of stationary phase was suggested. This method was developed by Kelvin and is well established in the problems of wave motions in ideal liquids.

This method is also applied in all publications listed in [2–21]. Unfortunately, a groundless application of it leads to incorrect determination of the asymptotic behavior of the free surface elevation in viscous liquids.

Some examples are given in Sect. 2 of such incorrect application of the stationary phase method to integrals J_1 and G_1 in [2, 4] that determine the shape of free surface arising from the initial δ-type elevation in the two- and three-dimensional cases of liquids of infinite depth. We also demonstrate in Sect. 2 some other errors in [2–4]. In Sect. 3 we develop the asymptotic expressions in the plane of variables x, t for the integral

$$\eta = \frac{S}{\pi} \int_0^\infty e^{-2\nu t k^2} \cos xk \cos t \sqrt{gk}\, dk \tag{1.1}$$

where x is the dimensional coordinate, t is the dimensional time, ν is the kinematic viscosity factor, and g is the gravity acceleration. This notation is used in Eq. (46) of [1]. Equation (1.1) was derived in [1] for a simplified description of the profile of waves η, caused by the elevation in the liquid surface S, concentrated at the origin of coordinates.

The asymptotic representation of η is developed on the paths $x = ct^\alpha$, where $c = \text{const} > 0$ and α is an arbitrary real number for $t \to \infty$ or $x \to \infty$. In particular, it is shown that the stationary phase method is applicable for $5/4 \leqslant \alpha < 2$. It is found in Sect. 4 that the asymptotic representation for the integrals J_1 and η are the same. This means that after all errors in the calculation of the asymptotic representation of J_1, committed in [2] have been corrected, the results do not agree with those obtained by means of the simplified approach to the Cauchy-Poisson problem put forward by Sretenskii as early as in 1941.

2. In [2–21] the stationary phase method is applied for the derivation of a number of integrals. A typical example is the last integral on p. 339 in the paper [2](*)

$$J_1 = -4 \int_{1,8}^{\infty} A \frac{h(a)}{3K^3} e^{\nu t h^2(a)(a^2-b^2-1)} \cos\left[\frac{h(a)}{K} - 4ab\frac{h^2(a)\,\omega}{K^2}\right] Kx\, da \tag{2.1}$$

$$A = \frac{a(a^2+b^2-1)}{(a^2-b^2-1)^2+4a^2b^2}, \quad h(a) = K\left(\frac{a^2}{4a^6+4a^4-1}\right)^{1/3} = Kh_1(a)$$

$$b = \left(a^2+1-\frac{1}{a}\right)^{1/2}, \quad K = \left(\frac{g}{\nu^2}\right)^{1/3}, \quad \omega = \frac{K\nu t}{2x}$$

where x is the dimensional coordinate, t is the dimensional time, ν is the kinematic viscosity factor, and g is the gravity acceleration. When calculating J_1 by the stationary phase method, the authors apply the following formula (cf. (5.4) in [2]):

$$\int_\alpha^\beta \psi(a) \cos z\varphi(a)\, da \sim \left[\frac{2\pi}{z\varphi''(\tau_0)}\right]^{1/2} \psi(\tau_0) \cos\left(\frac{\pi}{4} + z\varphi\ (\tau_0)\right) \tag{2.2}$$

Here τ_0 is the unique stationary point on (α, β) and $z \to \infty$. Assuming further that Kx is a large parameter and

$$h_1(a) \approx 4^{-1/3}a^{-4/3}, \quad h_1^2(a)(a^2-b^2-1) \approx -2^{-1/3}a^{-8/3}, \quad A \approx 1/2a \tag{2.3}$$

$$\varphi(a) = h_1(a) - 4abh_1^2(a)\,\omega \approx 4^{-1/3}a^{-4/3} - 4^{1/3}a^{-2/3}\omega$$

*) An obvious misprint in the original is corrected here.

finding $\tau_0 = \omega^{-3/2} / \sqrt{2}$, the following result is obtained:

$$J_1 \sim -\left(\frac{\pi\omega^2}{K^5 |x|}\right)^{1/2} e^{-2K^2 \nu\omega^4 t} \cos\left(\frac{\pi}{4} - K |x| \omega^2\right) \tag{2.4}$$

The remainder term in (2.4) is not written out but it is confirmed that the formula is valid for $\omega < 1/2$ (cf. pp. 339, 340 in [2]).

In connection with the above example of application of the stationary phase method in [2], let us make the following remarks:

1) The use of the formula (2.2) for the calculation of J_1 is not justified (*) because (2.2) is based on the assumption that both the amplitude $\psi(a)$ and phase $\varphi(a)$ are in dependent of the large parameter z when $z \to \infty$. The phase in J_1 does not depend on the large parameter $z = Kx$ when $z \to \infty$, if $\omega = \text{const}$, i.e. if $t = 2\omega z / K^2\nu$. Substituting this value into the amplitude of J_1, we find that $\psi(a)$ substantially depends on z ,

$$\psi(a) = A \frac{h(a)}{3K^3} \exp\left[2z\omega \frac{h^2(a)}{K^2}(a^2 - b^2 - 1)\right] \tag{2.5}$$

and it clear that the assumptions of (2.2) are not satisfied.

2) Equation (2.4) leads to incorrect notions about the asymptotic behavior of integral J_1. For instance, if the values of K, ν and ω are fixed, we have according to (2.4) for $t \to \infty$

$$J_1 = -\left(\frac{2\pi\omega^3}{K^6 \nu t}\right)^{1/2} e^{-2K^2 \nu\omega^4 t} \cos\left(\frac{\pi}{4} - K^2 \nu t \frac{\omega}{2}\right) \tag{2.6}$$

Actually, the true expression for J_1 , as will be shown in Sect. 4, is:

$$J_1 = \frac{\nu^2}{g^2 t} [1 + o(1)] \quad (t \to \infty) \tag{2.7}$$

3) Calculation of the asymptotic representation of J_1 is a fundamental step in [2] because in this paper integral J_1, – apart from a multiplier, determines the waves caused by the initial δ-type elevation of the free surface of a viscous liquid of infinite depth, when treated as a two-dimensional problem (cf. (5.11), (5.12) and (6.1) in [2]). All the ultimate results in this paper [2] are based on Eq. (2.4) and this leads to false ideas about the asymptotic behavior of waves.

Similarly, the stationary phase method is used in other cases of the amplitude substantially dependent on the large parameter when calculating the following integrals: (4.12) and (5.11) in [4]; (1.5) in [7]; (2.5) in [8]; (3.12) and (5.2) in [10]; (1.5) in [11]; (3) in [13]; (2.14) and (3.8) in [14]; (3.1) and (6.1) in [17]; (6) in [21].

In all the above quoted papers the stationary phase method is applied incorrectly and

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*) This error was pointed out by I. B. Simonenko and V. I. Iudovich. Let us quote the example given by Simonenko: formal application of the stationary phase method to the integral

$$I(k) = \int_0^\infty e^{-k\xi} \cos k\xi^2 \, d\xi \qquad (k \to \infty)$$

yields

$$I(k) \sim \left[\frac{\pi}{4k}\right]^{1/2} \cos\frac{\pi}{4} \qquad (k \to \infty)$$

whereas a simple estimate shows that

$$|I(k)| \leqslant \int_0^\infty e^{-k\xi} \, d\xi = \frac{1}{k} \qquad (k \to \infty)$$

this has resulted in wrong formulas and erroneous physical interpretations.

In the papers [4, 7–10] the stationary phase method is applied in a wrong manner and its formula is misinterpreted. The authors quote the well-known equation (cf. (5. 8) in [4] and (3. 13) in 10])

$$\int_{\alpha}^{\beta}\psi(a)\,e^{iz\varphi(a)}\,da= \tag{2.8}$$

$$=\left\{\left[\frac{2\pi}{z\,|\varphi''(\tau)|}\right]^{1/2}\psi(\tau)\exp\left[iz\varphi(\tau)+\frac{i\pi}{4}\operatorname{sign}\varphi''(\tau)\right]\right\}\left[1+O\left(\frac{1}{\sqrt{z}}\right)\right],\quad z\to\infty$$

(where τ is the stationary point $\varphi(a)$ and $\alpha<\tau<\beta$), but actually use an incorrect equation as follows:

$$\int_{\alpha}^{\beta}\psi(a)\cos z\varphi(a)\,da= \tag{2.9}$$

$$=\left\{\left[\frac{2\pi}{z\,|\varphi''(\tau)|}\right]^{1/2}\psi(\tau)\cos\left[z\varphi(\tau)+\frac{\pi}{4}\operatorname{sign}\varphi''(\tau)\right]\right\}\left[1+O\left(\frac{1}{\sqrt{z}}\right)\right],\quad z\to\infty$$

It is obvious that (2. 9) is wrong because it follows from it that the zeros of the first term of the asymptotics are (exactly !) the same as the zeros of the approximate function, if z is sufficiently great. (Thus, the zeros of the Bessel function $J_0(z)$ for $z\to\infty$ would be the same as the zeros of $\cos(z-\pi/4)$!).

Let us consider this concealed substitution of formulas in [4]; having applied the stationary phase method to integral G_1 (cf. (5. 9) in [4])

$$G_1=-\int_0^{\pi/2}\frac{\sqrt{g}\,\nu^2\tau^3}{8R^3\cos^3\theta\sqrt{\pi R\cos\theta}}\exp\frac{-\nu g^2\tau^5}{8R^4\cos^4\theta}\times \tag{2.10}$$

$$\times\cos\left(\frac{\pi}{4}-\frac{g\tau^2}{4R\cos\theta}\right)\left[1+O\left(\frac{1}{\sqrt{\lambda R\cos\theta}}\right)\right],\quad \lambda=(g/\nu^2)^{1/3},\quad \lambda R\to\infty$$

the authors write (cf. (5. 10) in [4]):

$$G_1=-\frac{\nu^2\tau^2}{8\sqrt{2}\,R^3}\exp\frac{-\nu g^2\tau^5}{8R^4}\cos\frac{g\tau^2}{4R}\times \tag{2.11}$$

$$\times\left[1+O\left(\frac{1}{\sqrt{\lambda R}}\right)+O\left(\sqrt{\frac{4R}{gt^2}}\right)\right],\quad \lambda R\to\infty,\quad \frac{gt^2}{4R}\to\infty$$

Obviously, even if integral (2. 10) makes sense and allows the use of the stationary phase method with the change from $O[(\lambda R\cos\theta)^{-1}]$ to $O[(\lambda R)^{-1}]$ because $\theta=0$ is the stationary point, Eq. (2. 11) can be derived from (2. 10) only be means of (2. 9).

Equation (2. 11) is of fundamental significance in [4] because, apart from a multiplying factor, G_1 determines the waves caused by an initial δ-type elevation of the free surface of a viscous liquid occupying half-space (cf. (6. 1) in [4]) (*)

$$\zeta(R,t)=\frac{Agt^2}{16\sqrt{2}\,R^3}\exp\frac{-\nu g^2t^5}{8R^4}\cos\frac{gt^2}{4R}\left[1+O\left(\frac{1}{\sqrt{\lambda R}}\right)+O\left(\sqrt{\frac{4R}{gt^2}}\right)\right] \tag{2.12}$$

From (2. 12) follows an obviously wrong conclusion that the zeros of the function ζ which describes the shape of the free surface of viscous liquid provided λR and $gt^2/4R$ are fairly great, are exactly the same as the zeros of the corresponding function for an ideal liquid, also when $gt^2/4R$ is great.

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*) In the original the notation is r instead of R.

It must be pointed out that the paper [4] contains also a number of other inaccuracies. Thus, Eq. (5.10) is derived assuming $\omega = \lambda\nu\tau/2R\cos\theta < 1/_2$ (cf. the text before (5.8) in [4]). Nevertheless, solutions (5.10) and (5.11) do not bear witness to this condition. Moreover, for the purpose of mechanical interpretation of Eq. (6.1) the motion of free surface of the liquid in a fixed space point (R = const) is analyzed for $t \to \infty$, although in this case it is ω which tends to infinity.

In the same paper the authors wrongly apply the equation of the Laplace method which produces asymptotics with respect to one parameter, while calculating integrals (5.1) that depend on two parameters. The obtained values of these integrals are wrong when $a_1 > 0$, $b_1 > 0$, which can be easily proved by comparing formulas 3.897, 8.253 and 8.254 in the reference book by I.S. Gradshtein and I. M. Ryzhik "Tables of Integrals, Sums, Series and Products" (1958). The same criticism applies also to [2].

3. Let us now derive correct asymptotic equations for integrals such as (1.1), (2.1) and others considered in [2-21]. We have already said in our introductory remarks that the author of [1] analyzed integral (1.1) when looking for a simplified description of the wave profile. By applying to it the method of "steady-state phases" for great values of $gt^2/4x$, he found that (approximately)

$$\eta = \frac{1}{2}\left(\frac{g}{\pi}\right)^{1/2}\frac{St}{x^{3/2}}\exp\frac{-\nu g^2 t^5}{8x^4}\cos\left(\frac{gt^2}{4x}-\frac{\pi}{4}\right) \tag{3.1}$$

Besides (3.1), he derived also an expression for small values of $\nu t/x^2$ (cf. Eq. (49) in [1])

$$\eta = \frac{Sgt^2}{2\pi x^2} \tag{3.2}$$

Equation (3.1) can be conveniently rewritten as

$$\eta = \eta_0 e^{-\varkappa} \qquad \eta_0 = \frac{St}{2x^{3/2}}\left(\frac{g}{\pi}\right)^{1/2}\cos\left(\frac{gt^2}{4x}-\frac{\pi}{4}\right), \qquad \varkappa = \frac{\nu g^2 t^5}{8x^4} \tag{3.3}$$

where η_0 is the solution of a corresponding problem for an ideal liquid, and factor $e^{-\varkappa}$ describes the effect of viscosity. Equation (3.3) plays substantial part in all problems considered in [2-21]. Formulas such as (3.2) are not derived in any of the papers [2-16] and [18-21].

Let us consider under what conditions Eqs. (3.1) and (3.2) describe asymptotic behavior of integral (1.1). Changing to dimensionless variables by means of

$$\xi = \lambda\eta, \quad S_1 = \lambda^2 S, \quad x_1 = \lambda x$$
$$u = \frac{k}{\lambda}, \quad \lambda = \left(\frac{g}{\nu^2}\right)^{1/3}, \quad t_1 = g^{2/3}\nu^{-1/3} t$$

and omitting subscripts in dimensionless variables, we have

$$\xi = \frac{S}{\pi}\int_0^\infty e^{-2tu^2}\cos xu \cos t\sqrt{u}\, du \tag{3.4}$$

Integral ξ is a function of two variables, x and t. On the other hand, the well-known equation of the stationary phase method

$$\int_\alpha^\beta \psi(a)\, e^{iz\varphi(a)}\, da = \left[\frac{2\pi}{z\,|\varphi''(\tau)|}\right]^{1/2}\psi(\tau)\, e^{i[z\varphi(\tau)+1/_4\pi\,\mathrm{sign}\,\varphi''(\tau)]} + O(z^{-1}) \tag{3.5}$$

(where τ is the unique stationary point $\varphi(a)$, $\alpha < \tau < \beta$) gives an asymptotic representation only with respect to one variable $z \to \infty$. It is, therefore, natural to ask on which paths in the plane of variables x, t formula (3.1) is truly valid. Let us consider the family of paths $x = ct^{\alpha}$, where α is a definite real number and c is a positive constant; we shall now investigate in this family the asymptotic behavior of integral ξ when $t \to \infty$ or $x \to \infty$.

Theorem 1. Let

$$t^{\alpha} / x = c = \text{const} > 0, \qquad \alpha < {}^5/_4, \qquad t > 0, \qquad x > 0$$

Then for $t \to \infty$

$$\xi = -\frac{2S}{\pi t^2}[1 + o(1)] \tag{3.6}$$

or in dimensional variables

$$\eta = -\frac{2S}{\pi g t^2}[1 + o(1)] \tag{3.7}$$

In this case the estimate (3.6) is uniform for all $\alpha \leqslant {}^5/_4 - \varepsilon$, where $\varepsilon > 0$.

Proof. Asymptotic formula (3.6) is derived by means of iterated integration by parts. First, setting $u = t^{-1/2}v^2$, we have from (3.4)

$$\xi = \frac{2S}{\pi\sqrt{t}} \int_0^\infty ve^{-2v^4} \cos(c^{-1}t^{\alpha-1/2} v^2) \cos t^{3/4} v \, dv$$

$$(\alpha < {}^5/_4, \quad \alpha - {}^1/_2 < {}^3/_4, \quad t \to \infty) \tag{3.8}$$

Let us now point out that under such conditions the third factor in the integrand has a greater effect on the asymptotic behavior of ξ, when $t \to \infty$, than the second factor. This determines the choice of procedure for the integration by parts. Assuming that $dy = \cos t^{3/4} v \, dv$, we have from (3.8)

$$\xi = \frac{2S}{\pi}[-J_1 + J_2 + J_3] \tag{3.9}$$

where

$$J_1 = t^{-5/4} \int_0^\infty e^{-2v^4} \cos(c^{-1} t^{\alpha-1/2} v^2) \sin t^{3/4} v \, dv$$

$$J_2 = t^{-5/4} \int_0^\infty 8v^4 e^{-2v^4} \cos(c^{-1}t^{\alpha-1/2} v^2) \sin t^{3/4} v \, dv$$

$$J_3 = t^{\alpha-7/4} \int_0^\infty 2c^{-1} v^2 e^{-2v^4} \sin(c^{-1}t^{\alpha-1/2} v^2) \sin t^{3/4} v \, dv$$

Integrating J_1 by parts ($dy = \sin t^{3/4} v dv$), we obtain

$$J_1 = \frac{1}{t^2}[1 - J_{11} - J_{12}] \tag{3.10}$$

where

$$J_{11} = 8\int_0^\infty v^3 e^{-2v^4} \cos(c^{-1}t^{\alpha-1/2} v^2) \cos t^{3/4} v \, dv$$

$$J_{12} = 2c^{-1}t^{\alpha-1/2} \int_0^\infty ve^{-2v^4} \sin(c^{-1}t^{\alpha-1/2} v^2) \cos t^{3/4} v \, dv$$

It can easily be shown that $J_{11} = o(1)$ when $t \to \infty$. Indeed, when J_{11} is integrated by parts, we have

$$J_{11} = -J_{111} + J_{112} + J_{113}$$

where

$$J_{111}=t^{-3/4}\int_0^\infty 24\, v^2 e^{-2v^4}\cos\left(c^{-1}t^{\alpha-1/2}\, v^2\right)\sin t^{3/4} v\, dv = O\left(t^{-3/4}\right)$$

$$J_{112}=t^{-3/4}\int_0^\infty 64\, v^6 e^{-2v^4}\cos\left(c^{-1}t^{\alpha-1/2}\, v^2\right)\sin t^{3/4} v\, dv = O\left(t^{-3/4}\right)$$

$$J_{113}=16\, c^{-1}\, t^{\alpha-5/4}\int_0^\infty v^4 e^{-2v^4}\sin\left(c^{-1}t^{\alpha-1/2}\, v^2\right)\sin t^{3/4} v\, dv = O\left(t^{\alpha-5/4}\right)$$

Hence, when $\alpha < 5/4$, we have

$$J_{11} = o\,(1), \qquad t\to\infty \tag{3.11}$$

To prove that $J_{12} = o\,(1)$ when $t\to\infty$ is rather more difficult. When J_{12} is integrated by parts, we obtain

$$J_{12} = -J_{121} + J_{122} - J_{123}$$

where

$$J_{121} = O\left(t^{\alpha-5/4}\right), \qquad J_{122} = O\left(t^{\alpha-5/4}\right), \qquad t\to\infty$$

$$J_{123}=4c^{-2}\, t^{2\alpha-7/4}\int_0^\infty v^2 e^{2v^4}\cos\left(c^{-2}t^{\alpha-1/2}\, v^2\right)\sin t^{3/4} v\, dv = O\left(t^{2\alpha-7/4}\right) \tag{3.12}$$

The first two estimates produce the required result because $\alpha < 5/4$, but the third estimate does not. However, it can be easily seen that with subsequent iterated integration by parts $n-2$ times, J_{123} will be given greater weight, namely

$$J_{123} = O\left(t^{3/4+(\alpha-5/4)\, n}\right), \qquad t\to\infty \tag{3.13}$$

Thus, if the required estimate is to be obtained, J_{123} must be integrated by parts until the following inequality is satisfied: $3/4 + n\,(\alpha - 5/4) < 0$

We have then from (3.12) and (3.13) that $J_{12} = o\,(1)$. From (3.10) and (3.11) we can now deduce that

$$J_1 = \frac{1}{t^2}\,[1 + o\,(1)], \qquad t\to\infty \tag{3.14}$$

In a very similar manner it can be found that

$$J_2 = o\,(t^{-2}), \qquad J_3 = o\,(t^{-2}), \qquad t\to\infty \tag{3.15}$$

The second estimate can be obtained only by integration by parts until the following condition is satisfied:

$$6/4 + n\,(\alpha - 5/4) < 0$$

Theorem 2. Let

$$t^5 / x^4 = c = \text{const} \qquad \text{or} \qquad x = c^{-1/4}\, t^{5/4}, \qquad x > 0, \qquad t > 0$$

Then for $x\to\infty$

$$\xi = \frac{S}{2\sqrt{\pi}}\,\frac{t e^{1/8 c}}{x^{3/2}}\cos\left(t^2/4x - \frac{\pi}{4}\right) + O\left(\frac{S}{x}\right) \tag{3.16}$$

or in dimensional variables

$$\eta = \frac{1}{2}\left(\frac{g}{\pi}\right)^{1/2}\frac{tSe^{-1/8 c}}{x^{3/2}}\cos\left(gt^2/4x - \frac{\pi}{4}\right) + O\left(\frac{S}{x}\right) \tag{3.17}$$

Proof. Assuming $ux^2 = t^2v^2$ and allowing for the condition $t^5 = cx^4$, we derive

$$\xi = \frac{S}{\pi}\,\frac{t^2}{x^2}\left[\int_0^\infty ve^{-2cv^4}\cos\frac{t^2}{x}\,(v^2 - v)\, dv + \int_0^\infty ve^{-2cv^4}\cos\frac{t^2}{x}\,(v^2+v)\, dv\right] \tag{3.18}$$

The stationary phase method may be applied to the first integral in (3.18) using Eq.

(3.5). From the conditions of the theorem we have

$$\psi(v) = ve^{-2cv^4}, \quad \varphi(v) = v^2 - v, \quad z = \frac{t^2}{x} = c^{1/4} t^{3/4} \to \infty, \quad \tau = 1/2$$

All conditions which must be satisfied if Eq. (3.5) is to be legitimately applied can be easily found. In particular, amplitude $\psi(v)$ and phase $\varphi(v)$ do not depend on the parameter $z \to \infty$. The second integral in (3.18) is easily estimated by means of integration by parts, and then (3.16) is obtained.

Equation (3.5) can be transferred also to the case when amplitude $\psi(a, z)$ depends on the parameter z but only to a slight extent. Possibility of such situation is pointed out in the book by J. J. Stoker (*). Obviously, some additional investigation is required in this case.

Theoerem 3. Let $t^\alpha = cx > 0$; α and n are such that

$$\frac{10n+12}{8n+9} \leqslant \alpha < \frac{10n+2}{8n+1}, \quad n \geqslant 0, \quad \frac{5}{4} < \alpha < 2, \qquad x > 0, \; c = \text{const}$$

Then, for $x \to \infty$

$$\xi = \frac{St}{2\pi^{1/2} x^{3/2}} \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \left(\frac{t^5}{8x^4}\right)^k \cos\left(\frac{t^2}{4x} - \frac{\pi}{4}\right) + o\left(\frac{S}{x}\right) \tag{3.19}$$

or in dimensional variables

$$\eta = \frac{1}{2}\left(\frac{g}{\pi}\right)^{1/2} \frac{St}{x^{3/2}} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right) \sum_{k=0}^{n} (-1)^k \frac{1}{k!}\left(\frac{\nu g^2 t^5}{8x^4}\right)^k + o\left(\frac{S}{x}\right) \tag{3.20}$$

Proof. Assuming $ut^{1/2} = v^2$, we transform (3.4) as follows:

$$\xi = I^+(0, 1) + I^-(0, 1) + I^+(1, \infty) + I^-(1, \infty)$$

$$I^\pm(a, b) = \frac{S}{\pi t^{3/2}} \int_a^b ve^{-2v^4} \cos k(v^2 \pm \varepsilon v)\, dv \tag{3.21}$$

$$ck = t^{\alpha - 1/2}, \quad \varepsilon = ct^{5/4 - \alpha}$$

Setting $du = k(2v \pm \varepsilon) \cos k(v^2 \pm \varepsilon v)\, dv$ and integrating by parts, we obtain the following estimates:

$$I^\pm(1, \infty) = O(St^{-\alpha}), \qquad t \to \infty \tag{3.22}$$

In order to compute $I^\pm(0,1)$ we expand the exponential term in the integrand into a Maclaurin series in the vicinity of 0. Using the asymptotic representation of Fresnel integrals, we obtain

$$I^+(0,1) = O(St^{-\alpha}), \qquad t \to \infty \tag{3.23}$$

$$I^-(0,1) = \frac{S\varepsilon}{2\pi\sqrt{t}} \left(\frac{\pi}{k}\right)^{1/2} \exp\frac{-\varepsilon^4}{8} \cos\left(\frac{ct^{2-\alpha}}{4} - \frac{\pi}{4}\right) + O(St^{-\alpha}) \tag{3.24}$$

Let us unify (3.21)–(3.24) and expand the exponential into a series in $1/8\, \varepsilon^4$. Leaving only those summands the order of which is no higher than that of the remainder, we arrive at (3.19).

Theorem 4. Let

$$x = ct^\alpha, \quad \alpha > 2, \quad t > 0, \quad x > 0 \;\text{ or }\; t = c = \text{const} > 0$$

Then, for $x \to \infty$,

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*) Editorial Note. J. J. Stoker, Water Waves: the Mathematical Theory with Applications. Interscience Publishers, Inc., New York, 1957.

$$\xi = \frac{St^2}{2\pi x^2}[1 + o(1)] \tag{3.25}$$

or in dimensional variables

$$\eta = \frac{Sgt^2}{2\pi x^2}[1 + o(1)] \tag{3.26}$$

As in the case of Theorem 1, the proof consists of the iterated integration by parts. In the present case we set:

$$v \cos(ct^{\alpha-1/2} v^2)\, dv = dy \text{ for odd-numbered steps}$$
$$v \sin(ct^{\alpha-1/2} v^2)\, dv = dy \text{ for even-numbered steps}$$

It may seem at first that estimate (3.25) covers a wider range of values of $\alpha > 5/4$. However, the remainder can be estimated only for $\alpha > 2$. To prove our theorem in the second case ($t =$ const, $x \to \infty$) it is necessary to change in the above substitution formulas from ct^α to x and from t to c.

4. Let us now analyze the integrals appearing in [2–21]; we shall take as an example integral (2.1) and prove that the asymptotic behavior of (2.1) is determined by expression (3.7) if $\alpha < 5/4$ and by (3.26) if $\alpha > 2$.

Setting $u = (2a^2)^{-2/3}$ and changing to dimensionless variables x and t, we obtain from (2.1)

$$J_1 = -\frac{1}{2K^2}(J_2 + J_3), \qquad B = (1{,}8\sqrt{2})^{-4/3} \tag{4.1}$$

$$J_{2,3} = \int_0^B \varphi_1(u)\, e^{-2tu^2\varphi_2(u)} \begin{matrix}\cos\\ \sin\end{matrix} xu\, \varphi_3(u) \begin{matrix}\cos\\ \sin\end{matrix} t\sqrt{u}\, \varphi_4(u)\, du$$

The functions φ_k in the above formula can be written out as series

$$\varphi_k(u) = \sum_{n=0}^{\infty} a_{kn}(u^{3/4})^n \qquad (k = 1, 2, 3, 4) \tag{4.2}$$

converging in the vicinity of the point $u = 0$, while $\varphi_k(0) = 1$. The exponential asymptotics of integrals J_2 and J_3 on the paths $x = ct^\alpha$ ($\alpha < 5/4$, $\alpha > 2$) is produced by the vicinity of the point $u = 0$, because it is easy to estimate the upper limit of the corresponding integrals on the segment $[\varepsilon, B]$ and such estimate takes the form of $\exp(-mt)$, $m > 0$ when $t \to \infty$. On the segment $[0, \varepsilon]$ the integrand in J_2 is practically the same as the integrand in Sretenskii's integral, provided ε is sufficiently small. It can be easily shown that Theorems 1–4 of Sect. 3 apply to integral J_2 in which $B = \varepsilon$ and ε is sufficiently small. It is necessary to use substitution $u\varphi_2^2 = t^{-1/2} v^2$ instead of $u = t^{-1/2} v^2$. In a similar manner (cf. Sect. 3) it can be shown that

$$J_3 = o(t^{-2}), \quad t \to \infty \text{ on the paths } x = ct^\alpha, \quad \alpha < 5/4$$
$$J_3 = o(t^2 / x^2), \quad x \to \infty \text{ on the paths } x = ct^\alpha, \quad \alpha > 2$$

It follows that in dimensioned variables

$$J_1 = \frac{\nu^2}{g^2 t^2}[1 + o(1)], \quad t \to \infty \text{ on the paths } x = ct^\alpha, \quad \alpha < 5/4$$

$$J_1 = -\frac{\nu^2 t^2}{4x^2}[1 + o(1)], \text{ on the paths } x = ct^\alpha, \ \alpha > 2, \quad x \to \infty$$

Note 1. As has been already said in Sect. 2, integral (2.1) in [2] was actually calculated on the basis of assumptions (2.3). In this case, functions $\varphi_k(u) \equiv 1$ ($k = 1, 2, 3, 4$) and the integrand in J_1 is (exactly !) the same as the integrand in Sretenskii's integral. This means, that after all errors in calculating J_1 are put right, the results will not agree with those derived by means of the simplified consideration of the Cauchy-Poisson problem.

Note 2. The arguments in Sect. 3 that led to the derivation of the asymptotics on the paths $x = ct^\alpha$ can be easily applied also to the three-dimensional problem of waves caused by a stimulus concentrated at the point of origin of coordinates on the surface of a viscous liquid in half-space or layer.

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ANALOGY BETWEEN EQUATIONS OF PLANE FILTRATION AND EQUATIONS OF LONGITUDINAL SHEAR OF NONLINEARLY ELASTIC AND PLASTIC SOLIDS

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There is a simple analogy between plane problems of nonlinear filtration and problems of longitudinal shear of nonlinearly elastic and plastic solids which makes it possible to transfer results and problem formulations from one field to the other. We formulate this analogy in explicit form (Sect. 1), consider some examples and consequencies (Sect. 2), and justify a variational principle for the equations of nonlinear filtration, which together with the maximum principle yield estimates for the integral characteristics of a filtration stream (Sect. 3).

1. 1°. The system of equations of plane nonlinear filtration of an incompressible fluid consists of the filtration law equations and the continuity equation [1, 2]

$$\operatorname{grad} H = -\Phi(w)\,\mathbf{w}/w, \qquad \operatorname{div}\mathbf{w} = 0 \tag{1.1}$$